Simple method for calculating the Casimir energy for sphere

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A simple method for calculating the Casimir energy for a sphere is developed which is based on a direct mode summation and counter integration in a complex plane of eigenfrequencies. The method uses only classical equations determining the eigenfrequencies of the quantum field under consideration. Efficiency of this approach is demonstrated by calculation of the Casimir energy for a perfectly conducting spherical shell and fora massless scalar field obeying the Dirichlet and Neumann boundary conditions on sphere. The possibility of rationalizing the removal of divergences in this problem as a renormalization of both the energy and the radius of the sphere is discussed.

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I. INTRODUCTION

The Casimir effect attracts the attention of investigators during last half of a century. More generally the Casimir effect can be defined as an influence of the boundness of the configuration space on the physical characteristics of the quantum field system (its energy, forces and momentum of forces acting on the boundaries and so on).

This problem arises in different areas of theoretical physics: in quantum electrodynamics (attractive force between two uncharged conducting plates calculated by Casimir in 1948), in the theory of elementary particles (the bag models of hadrons treat the energy of quark and gluon fields located inside hadrons), in current cosmology, in physics of condensed matter (elucidation of the physical origin of sonoluminescence) and so on.

When considering the Casimir effect different methods are used: Green function formalism [1], stress-tensor method [2], multiple scattering expansion [3], zeta regularization technique [4], heat-kernel series [5], direct mode summation with counter integration [6,7]. Physical interpretations proposed in different approaches to calculating the Casimir effect are distinct. For example, the Casimir forces between two uncharged conducting plates can be treated both as the macroscopic manifestation of the van der Waals forces and as an effect

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of zero point oscillations of vacuum electromagnetic field. In this situation it is worthwhile to separate in particular calculations an invariant, with respect to physical interpretation, "kernel" which gives the final result.

In all the approaches to calculation of the Casimir effect a vague point is the procedure of unique separation and subsequent removal of the divergences. The lack of universal mathematically rigorous prescription for this purpose leads in some problems to different results when different methods are applied (for example, calculation of the Casimir energy for scalar massless field defined on the plane and subjected to the Dirichlet boundary conditions on a circle [8]).

With regard to all this, the most simple, from mathematical point of view, methods of calculation of the Casimir effect have an obvious advantage because they right away allow one to reveal the difficulties generated by divergences. One of such methods is the direct summation of eigenfrequencies of quantum field system by making use of counter integration in complex frequency plane. For the first time this approach was proposed as a simple, in comparison with quantum field theory formalism, method of calculation of the van der Waals forces between dielectrics [9]. Further it was widely used in other problems [6,7,10,11].

The main goal of this paper is to show the simplicity and efficiency of the direct mode summation by counter integration when calculating the Casimir energy for such a difficult boundary as sphere. This approach is completely based on using the classical frequencies of quantum field system concerned, and the main tool employed is the Cauchy theorem from complex analysis. In this approach the Casimir energy for perfectly conducting and infinitely thin spherical shell will be calculated. Then the Casimir energy of scalar massless field subjected to the Dirichlet or Neumann boundary conditions on sphere will be also derived. As far as we know the last problem (the Neumann boundary conditions) has not been considered in other approaches yet. Unlike other authors we propose to interpret the removal of divergences when calculating the Casimir effect for sphere as the renormalization not only of the energy but also of the radius of the sphere.

The organization of the paper is as follows. In Sec. II we consider the vacuum energy of the electromagnetic field inside and outside the perfectly conducting spherical shell. In Sec. III the Casimir energy of massless scalar field obeying the Dirichlet or Neumann boundary conditions on sphere is calculated. Concluding remarks and discussion of necessity for renormalization of the sphere radius, in addition to the energy renormalization, are presented in Sec. IV.

II. VACUUM ELECTROMAGNETIC FIELD INSIDE AND OUTSIDE THE PERFECTLY CONDUCTING SPHERICAL SHELL

The starting point of our approach is the following definition of the Casimir energy

$$E = \frac{1}{2} \sum_s (\omega_s - \bar{\omega}_s)\,. \tag{2.1}$$

Here ω_s are the eigenfrequencies of the system under consideration, and $\bar{\omega}_s$ are those of the same system, when the parameters determining its boundaries take on some limiting values. We need equations for the oscillation frequencies of the electromagnetic field inside

and outside the perfectly conducting sphere with radius a. There are two modes of oscillations: transverse-electric modes and transverse magnetic ones (TE-modes and TM-modes, respectively). The eigenfrequencies of the TE-modes are defined by the equations [12]

$$j_l(\omega a) = 0, \tag{2.2}$$
$$h_l^{(1)}(\omega a) = 0, \tag{2.3}$$

and the eigenfrequencies for the TM-modes are given by

$$\frac{d}{dr}\left[r j_l(\omega r)\right]\Big|_{r=a} = 0, \tag{2.4}$$
$$\frac{d}{dr}\left[r h_l^{(1)}(\omega r)\right]\Big|_{r=a} = 0. \tag{2.5}$$

In formulae (2.2)-(2.5) $j_l(z)$ and $h_l^{(1)}(z)$ are the spherical Bessel functions [13]

$$j_l(z) = \sqrt{\frac{\pi}{2z}}J_{l+1/2}(z), \quad h_l^{(1)}(z) = \sqrt{\frac{\pi}{2z}}H_{l+1/2}^{(1)}(z), \tag{2.6}$$

and $l = 1, 2, \ldots$. Only positive roots of these equations $\omega_{nl} > 0$, $n = 1, 2, \ldots$ should be considered. Equations (2.2) and (2.4) specify the frequencies of the electromagnetic oscillations inside the sphere and Eqs. (2.3) and (2.5) give the frequencies outside the sphere [12].

In the case of spherical boundary the sum $\sum_s$ in (2.1) can be written as

$$\frac{1}{2}\sum_s \omega_s = \frac{1}{2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{n=1}^{\infty}\omega_{nl} = \sum_{l=1}^{\infty}(l+1/2)S_l, \tag{2.7}$$

Where $S_l = \sum_{n=1}^{\infty}\omega_{nl}$, and each frequency equation (2.2)–(2.5) generates its partial sum S_l^{α}, $\alpha = 1, \ldots, 4$.

For the partial sums $S_l^{(\alpha)}$ we use integral representation that follows from the Cauchy theorem [14]

$$S_l^{(\alpha)} = \frac{1}{2\pi i}\oint_C dz\, z\frac{d}{dz}\ln f^{(\alpha)}(z, a). \tag{2.8}$$

Here $f^{(\alpha)}(z, a)$ are the functions defining the frequency equations (2.2)–(2.5) in the form

$$f^{(\alpha)}(\omega, a) = 0, \quad \alpha = 1, 2, 3, 4. \tag{2.9}$$

The counter C encloses counterclockwise positive roots of these equations. Taking into account the position of the roots on real axis one can deform the counter C in such a way that it will consist of the segment $[-i\Lambda,\ i\Lambda]$ of the imaginary axis and a semicircle of radius Λ with $\Lambda \to \infty$ in the right half-plane. When Λ is fixed, the counter integral (2.8) gives the regularized value of corresponding frequency sum (it sums up the finite number of the roots of the frequency equation (2.8) that lie inside the counter).

From physical considerations it is clear that for negative values of the argument ω the functions $f^{(\alpha)}(\omega, a)$ have to be defined by a condition

$$f^{(\alpha)}(-\omega, a) = f^{(\alpha)}(\omega, a), \quad \omega > 0. \tag{2.10}$$

This can be achieved, for example, by introducing a finite photon mass which is equated to zero at the end of the calculations. Separating the contributions of different parts of the counter C, we can rewrite formula (2.8) as

$$S_l^{(\alpha)} = -\frac{1}{2\pi} \int\limits_{-\Lambda}^{+\Lambda} dy\, y \frac{d}{dy} \ln f^{(\alpha)}(iy, a) + \frac{1}{2\pi i} \int\limits_{C_\Lambda} z\, d \ln f^{(\alpha)}(z, \alpha). \tag{2.11}$$

Here C_Λ is the semicircle of radius Λ introduced above. In the first term on the right-hand side of (2.11) we can integrate by parts, the nonintegral term being omitted in view of (2.10). On the other hand this term can be removed by a subtraction which we shall discuss further.

In accordance with the definition (2.1) in order to obtain a finite (observable) value of the Casimir energy it is necessary to perform the subtraction. As usual, we shall subtract the contribution of the Minkowski space that corresponds to the limit $a = \infty$ in Eq. (2.11). Letting $\bar{S}_l^{(\alpha)}$ represent the value of the partial sum $S_l^{(\alpha)}$ which is to be subtracted from (2.11) we get

$$\bar{S}_l^{(\alpha)} = \frac{1}{2\pi} \int\limits_{-\Lambda}^{+\Lambda} dy\, \ln f^{(\alpha)}(iy, a \to \infty) + \frac{1}{2\pi i} \int\limits_{C'_\Lambda} z\, d \ln f^{(\alpha)}(z, a \to \infty). \tag{2.12}$$

In view of an oscillating character of the function $f^{(\alpha)}(z, a)$ (see below) we have on the semicircle C_Λ

$$\lim_{a \to \infty} f^{(\alpha)}(z, a) = f^{(\alpha)}(z, a) \tag{2.13}$$

Thus, for the difference $S_l^{(\alpha)} - \bar{S}_l^{(\alpha)}$ in the definition of the Casimir energy (2.1), we obtain

$$S_l^{(\alpha)} - \bar{S}_l^{(\alpha)} = \frac{1}{\pi} \int\limits_0^\infty dy\, \ln \left[\frac{f^{(\alpha)}(iy, a)}{f^{(\alpha)}(iy, a \to \infty)} \right]. \tag{2.14}$$

Here again the property (2.10) has been used. Now we proceed to substituting into Eq. (2.14) the concrete expressions for the functions $f^{(\alpha)}$ defined by frequency equations (2.2)–(2.5). From Eq. (2.2) we have

$$\frac{f^{(1)}(iy, a)}{f^{(1)}(iy, a \to \infty)} = \frac{J_\nu(iya)}{\lim\limits_{a \to \infty} J_\nu(iya)} = \frac{I_\nu(ay)}{\lim\limits_{a \to \infty} I_\nu(ay)} = \sqrt{2\pi a y}\, e^{-ay} I_\nu(ay), \tag{2.15}$$

where $\nu = l + 1/2$, and $I_\nu(z)$ is the modified Bessel function $J_\nu(iz) = i^\nu I_\nu(z)$. We have used here the asymptotics of the function $I_\nu(z)$ for fixed value of ν and large z [13]

$$I_\nu(z) \simeq \frac{e^z}{\sqrt{2\pi z}}. \tag{2.16}$$

From frequency equation (2.3) it follows that

$$\frac{f^{(2)}(iy, a)}{f^{(2)}(iy, a \to \infty)} = \frac{H_\nu^{(1)}(iay)}{\lim\limits_{a\to\infty} H_\nu^{(1)}(iay)}, \tag{2.17}$$

where $H_\nu^{(1)} = J_\nu(z) + iN_\nu(z)$ is the Hankel function of the first kind. Using the modified Bessel functions $K_\nu(z) = (\pi/2)i^{\nu+1}H_\nu^{(1)}(iz)$ we rewrite Eq. (2.17) as

$$\frac{f^{(2)}(iy, a)}{f^{(2)}(iy, a \to \infty)} = \frac{K_\nu(ay)}{\lim\limits_{a\to\infty} K_\nu(ay)} = \sqrt{\frac{2ay}{\pi}} e^{ay} K_\nu(ay). \tag{2.18}$$

Here we employed the asymptotics of the function $K_\nu(z)$ for large z and fixed ν [13]

$$K_\nu(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z}. \tag{2.19}$$

Thus, the TE-modes give the following contribution to Eq. (2.1)

$$\sum_{\alpha=1}^{2} (S_l^{(\alpha)} - \bar{S}_l^{(\alpha)}) = \frac{1}{\pi} \int\limits_0^\infty dy \, \ln\left[2ay I_\nu(ay) K_\nu(ay)\right]. \tag{2.20}$$

In the same way we deduce from the frequency equation (2.4)

$$\begin{aligned} \frac{f^{(3)}(iy, a)}{f^{(3)}(iy, a \to \infty)} &= \frac{J_\nu(iya)/2 + iya J_\nu'(iya)}{\lim\limits_{a\to\infty}[J_\nu(iya)/2 + iya J_\nu'(iya)]} \\ &= \frac{I_\nu(ya)/2 + ya I_\nu'(ya)}{\lim\limits_{a\to\infty}[I_\nu(ya)/2 + ya I_\nu'(ya)]} \end{aligned} \tag{2.21}$$

The prime over the Bessel function means the differentiation with respect to its argument. From (2.16) it follows that

$$\lim_{a\to\infty} [I_\nu(ay)/2 + ay I_\nu'(ay)] = \sqrt{\frac{ay}{2\pi}} e^{ay}. \tag{2.22}$$

In view of this, formula (2.21) assumes the form

$$\frac{f^{(3)}(iy, a)}{f^{(3)}(iy, a \to \infty)} = \sqrt{\frac{2\pi}{ay}} e^{-ay} \left[I_\nu(ay)/2 + ay\, I_\nu'(ay)\right]. \tag{2.23}$$

In the same way, for the frequency equation (2.5) we obtain

$$
\begin{aligned}
&\frac{f^{(4)}(iy,a)}{f^{(4)}(iy,a\to\infty)} \\
&= \frac{H_\nu^{(1)}(iya)/2 + iayH_\nu^{(1)'}(iay)}{\lim\limits_{a\to\infty}\left[H_\nu^{(1)}(iay)/2 + iayH_\nu^{(1)'}(iay)\right]} \\
&= \frac{K_\nu(ay)/2 + ayK'_\nu(ay)}{\lim\limits_{a\to\infty}\left[K_\nu(ay)/2 + ayK'_\nu(ay)\right]} \\
&= -\sqrt{\frac{2}{\pi ya}}e^{ay}\left[K_\nu(ay)/2 + ayK'_\nu(ay)\right]. \qquad (2.24)
\end{aligned}
$$

Summing up Eq. (2.23) and (2.24), we arrive at the contribution of the TM-modes

$$
\begin{aligned}
&\sum_{\alpha=3}^{4}(S_l^{(\alpha)} - \bar{S}_l^{(\alpha)}) \\
&= \frac{1}{\pi}\int\limits_0^\infty dy\,\ln\left\{-\frac{2}{ay}\left[\frac{1}{2}I_\nu(ay) + ayI'_\nu(ay)\right]\right. \\
&\quad\left.\times\left[\frac{1}{2}K_\nu(ay) + ayK'_\nu(ay)\right]\right\}. \qquad (2.25)
\end{aligned}
$$

Finally for the Casimir energy (2.1) we obtain from (2.7), (2.20) and (2.25)

$$
E = \frac{1}{\pi a}\sum_{l=1}^{\infty}\left(l+\frac{1}{2}\right)\int\limits_0^\infty dy\,\ln\left[1 - (\sigma'_l(y))^2\right], \qquad (2.26)
$$

where the notation

$$
\begin{aligned}
1 - (\sigma'_l(y))^2 &= -4I_\nu(y)K_\nu(y) \\
&\quad\times\left[\frac{1}{2}I_\nu(y) + yI'_\nu(y)\right]\left[\frac{1}{2}K_\nu(y) + yK'_\nu(y)\right]
\end{aligned}
$$

is introduced. Using the value of the Wronskian of the modified Bessel functions $I_\nu(y)$ and $K_\nu(y)$ [13]

$$
I_\nu(y)K'_\nu(y) - I'_\nu(y)K_\nu(y) = -\frac{1}{y}
$$

one can show that

$$
\sigma_l(y) = yI_\nu(y)K_\nu(y), \quad \nu = l + 1/2.
$$

The integral in (2.26) converges. This follows from the asymptotics of $\sigma'_l(y)$ for large y and fixed $\nu = l + 1/2$ [13]

$$
\sigma'_l(y) \simeq -\frac{1}{2y^2}\left[1 - \frac{1}{2}\frac{4\nu^2 - 1}{(2y)^2} + \ldots\right]. \qquad (2.27)
$$

Formula (2.26) coincides with Eq. (5.1) in paper [1], on the condition that the cut-off factor in the last equation is omitted and the integration by parts is performed. To carry out the summation with respect to l in (2.26) one needs the behavior of the integral

$$Q_l = \frac{l+1/2}{\pi} \int\limits_0^\infty dy \, \ln \left[1 - (\sigma_l'(y))^2\right] \tag{2.28}$$

at large l. Applying the uniform with respect to z asymptotics for the modified Bessel functions at large ν [1,13]

$$I_\nu(\nu z) K_\nu(\nu z) \simeq \frac{1}{2\nu} \frac{1}{(1+z^2)^{1/2}}, \tag{2.29}$$

we obtain from (2.28)

$$\begin{aligned} Q_l &\simeq \frac{\nu^2}{\pi} \int\limits_0^\infty dz \, \ln \left[1 - \frac{1}{4\nu^2 (1-z^2)^3}\right] \\ &\simeq -\frac{1}{4\pi} \int\limits_0^\infty \frac{dz}{(1+z^2)^3} = -\frac{3}{64}, \quad l \to \infty. \end{aligned} \tag{2.30}$$

Thus, the sum (2.26) at large l diverges as $\sum_{l=1}^\infty (l+1/2)^0$. To determine the finite value for this sum we rewrite (2.26) in the following way

$$\begin{aligned} E &= \frac{1}{a} \sum_{l=1}^\infty \left[Q_l + \frac{3}{64} - \frac{3}{64}\right] \\ &= \frac{1}{a} \sum_{l=1}^\infty \bar{Q}_l - \frac{3}{64a} \sum_{l=1}^\infty \left(l + \frac{1}{2}\right)^0, \end{aligned} \tag{2.31}$$

where

$$\bar{Q}_l = Q_l + 3/64. \tag{2.32}$$

The sum $\sum_{l=1}^\infty \bar{Q}_l$ converges because at large l

$$\bar{Q}_l = -\frac{9}{16384\nu^2} + \mathcal{O}(\nu^{-4}). \tag{2.33}$$

The last divergent sum in (2.31) can be defined by using the Hurwitz zeta function [14]

$$\zeta(z,q) = \sum_{n=0}^\infty \frac{1}{(q+n)^z}, \tag{2.34}$$

which at $q = 1/2$ is related with the Riemann ζ-function [15]

$$\zeta(z, 1/2) = (2^z - 1)\zeta(z). \tag{2.35}$$

From (2.34) it follows that

$$-\frac{3}{64a} \sum_{l=1}^\infty \left(l + \frac{1}{2}\right)^0 = -\frac{3}{64a} \left(\zeta(0,1/2) - 1\right) = \frac{3}{64a} \tag{2.36}$$

since $\zeta(0, 1/2) = 0$.

Finally for the Casimir energy we obtain

$$E = \frac{1}{a}\sum_{l=1}^{\infty}\bar{Q}_l + \frac{3}{64a}, \tag{2.37}$$

where $\bar{Q}_l$ is defined in (2.32) and (2.28). The sum $\sum_l \bar{Q}_l$ in the right-hand side of (2.37) can be estimated with allowance for the asymptotics (2.33)

$$\begin{aligned}\sum_{l=1}^{\infty}\bar{Q}_l &\simeq -\frac{9}{16384}\sum_{l=1}^{\infty}\frac{1}{(l+1/2)^2} = -\frac{9}{2^{14}}[\zeta(2,1/2)-4] \\ &= -\frac{9}{2^{14}}\left(\frac{\pi^2}{2}-4\right) = -0.000514\ldots .\end{aligned} \tag{2.38}$$

Thus, the main contribution to (2.37) is given by the second term and to a good approximation one can put for the Casimir energy

$$E \simeq \frac{3}{64a} = \frac{1}{a}0.046875. \tag{2.39}$$

Taking into account (2.38) we get instead of (2.39)

$$E \simeq \frac{1}{a}0.046361\ldots .$$

With greater accuracy this energy has been calculated in [1].

III. SCALAR FIELD OBEYING THE DIRICHLET OR NEUMANN BOUNDARY CONDITIONS ON SPHERE

The suggested method can be easily applied to the calculation of the Casimir energy of a massless scalar field subjected to boundary conditions on a sphere. Let us first consider the Dirichlet boundary conditions. In this case the eigenfrequencies are given by Eqs. (2.2) and (2.3) with $l = 0, 1, 2\ldots$. From (2.1), (2.7) and (2.20) we obtain

$$\begin{aligned}E^{(\mathcal{D})} &= \sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\frac{1}{\pi}\int_0^{\infty} dy\, \ln[2ayI_\nu(ay)K_\nu(ay)] \\ &= \frac{1}{a}\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\frac{1}{\pi}\int_0^{\infty} dy\, \ln[2yI_\nu(y)K_\nu(y)], \\ \nu &= l+1/2.\end{aligned} \tag{3.1}$$

It is worth comparing this expression with Eq. (3.5), derived in paper [16] by the Green function method. The essential advantage of our approaches is absence in the integrand of the constant terms which lead to the divergences (so-called contact terms). For large ν the integral in (3.1) behaves as follows

$$
\begin{aligned}
Q_l &\equiv \frac{l+1/2}{\pi}\int\limits_0^\infty dy\,\ln[2yI_\nu(y)K_\nu(y)] \\
&\simeq -\frac{\nu^2}{2}-\frac{1}{128}+\frac{35}{32768\nu^2}+\mathcal{O}(\nu^{-3}).
\end{aligned}
\tag{3.2}
$$

The first two terms in (3.2) give rise to divergences when summing up with respect to l in (3.1). These divergences can be removed as in the previous section by means of the Hurwitz ζ-function (2.34). On this purpose we rewrite (3.1)

$$
\begin{aligned}
E^{(\mathcal{D})} &= \frac{1}{a}\sum_{l=0}^{\infty} Q_l \\
&= \frac{1}{a}\sum_{l=0}^{\infty}\left(Q_l+\frac{\nu^2}{2}+\frac{1}{128}\right) \\
&\quad -\frac{1}{2a}\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)^2-\frac{1}{128a}\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)^0 .
\end{aligned}
\tag{3.3}
$$

Here we have added and subtracted under the sum sign two first terms of asymptotic expansion (3.2). Taking into account (2.34) we obtain

$$
\begin{aligned}
E^{(\mathcal{D})} &= \frac{1}{a}\sum_{l=0}^{\infty}\bar{Q}_l-\frac{1}{2a}\zeta\left(-2,\frac{1}{2}\right)-\frac{1}{128a}\zeta\left(0,\frac{1}{2}\right) \\
&= \frac{1}{a}\sum_{l=0}^{\infty}\bar{Q}_l,
\end{aligned}
\tag{3.4}
$$

where

$$
\bar{Q}_l = Q_l+\frac{1}{2}\left(l+\frac{1}{2}\right)^2+\frac{1}{128}. \tag{3.5}
$$

We derived the last equality in (3.4) bearing in mind that $\zeta(-2,1/2)=0$ and $\zeta(0,1/2)=0$. By virtue of the asymptotics (3.2) the last sum in (3.4) obviously converges.

With increase of ν, the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ are rapidly approaching their uniform asymptotics. That is why even for comparatively small values of l we can assume with allowance for (3.2) and (3.5)

$$
\bar{Q}_l \simeq \bar{Q}_l^{asym} = \frac{35}{32768\nu^2}. \tag{3.6}
$$

This simplifies the numerical calculations considerably. It is important to note that direct calculation of the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ with desired accuracy for all z and at large ν is a technically difficult problem [17]. It is especially concerns the product $I_\nu(z)K_\nu(z)$ encountered in the integrand of (3.2).

Table I demonstrates the applicability of the formula (3.6). Calculating $\bar{Q}_l$ for $l \le 3$ by means of numerical integration of the expressions including the product $I_\nu(y)K_\nu(y)$ (see Eqs. (3.5) and (3.2)) and using the asymptotic formula (3.6) for $l>3$ we obtain

$$E^{(\mathcal{D})} = \frac{1}{a}\,0.002819\ldots. \tag{3.7}$$

With greater accuracy the Casimir energy for spherical conducting shell was calculated in [16] by making use of the Green function technique.

Now we proceed to the consideration of the energy of a scalar field obeying the Neumann boundary conditions on the sphere. The eigenfrequencies inside and outside the sphere are defined now by equations

$$\frac{d}{dr}\left[j_l(\omega r)\right]|_{r=a} = 0, \tag{3.8}$$

$$\frac{d}{dr}\left[h_l^{(1)}(\omega r)\right]\Big|_{r=a} = 0, \tag{3.9}$$

where l takes values $0, 1, 2, \ldots$. By analogy with Eq. (2.25) we derive

$$\begin{aligned} E^{(\mathcal{N})} &= \\ &= \sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\frac{1}{\pi}\int\limits_0^{\infty} dy\, \ln\left\{-\frac{2}{ay}\left[\frac{1}{2}I_\nu(ay) - ayI'_\nu(ay)\right]\right. \\ &\quad \times \left.\left[\frac{1}{2}K_\nu(ay) - ayK'_\nu(ay)\right]\right\}. \end{aligned} \tag{3.10}$$

As in the case of electromagnetic field it is convenient at first to consider the sum $E^{(\mathcal{D})}+E^{(\mathcal{N})}$. From (3.1) and (3.10) it follows that

$$\begin{aligned} &E^{(\mathcal{D})} + E^{(\mathcal{N})} \\ &= \frac{1}{\pi a}\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\int\limits_0^{\infty} dy\, \ln\left[1-(\mu_l(y))^2\right], \end{aligned} \tag{3.11}$$

where the notation

$$\begin{aligned} 1-(\mu_l(y))^2 \equiv -4I_\nu(y)K_\nu(y)&\left[\frac{1}{2}I_\nu(y) - yI'_\nu(y)\right] \\ \times &\left[\frac{1}{2}K_\nu(y) - yK'_\nu(y)\right] \end{aligned} \tag{3.12}$$

is introduced. It is easy to show that

$$\mu_l(y) = y^2\frac{d}{dy}\left(\frac{1}{y}I_\nu(y)K_\nu(y)\right). \tag{3.13}$$

The integral in (3.11) converges since for large y and fixed l the following estimation

$$\mu_l(y) \simeq -\frac{1}{y}, \quad y\to\infty \tag{3.14}$$

holds.

The convergence of the sum in (3.11) is determined by the behavior at large l of the expression

$$P_l = \frac{l+1/2}{\pi}\int\limits_0^\infty dy \ln\left[1-(\mu(y))^2\right]. \tag{3.15}$$

Using the uniform asymptotics of the modified Bessel functions [13], we obtain

$$P_l \simeq -\frac{19}{64} - \frac{153}{16384\nu^2} + \mathcal{O}(\nu^{-3}). \tag{3.16}$$

The first term in this expression leads to a divergence in (3.11) when summing with respect to l. We again overcome this difficulty using the Hurwitz zeta function

$$\begin{aligned} E^{(\mathcal{D})} + E^{(\mathcal{N})} &\equiv \frac{1}{a}\sum_{l=0}^{\infty} P_l = \frac{1}{a}\sum_{l=0}^{\infty}\bar{P}_l - \frac{19}{64a}\zeta\left(0,\frac{1}{2}\right) \\ &= \frac{1}{a}\sum_{l=0}^{\infty}\bar{P}_l, \end{aligned} \tag{3.17}$$

where $\bar{P}_l$ is the renormalized value of P_l

$$\bar{P}_l = P_l + \frac{19}{64} \tag{3.18}$$

When calculating the last sum in (3.17) we can again assume

$$\bar{P}_l \simeq \bar{P}_l^{asym} = -\frac{153}{16384\nu^2}, \quad \nu = l+1/2. \tag{3.19}$$

Numerical calculation shows (see Table II) that with increasing l P_l approaches rapidly to its asymptotic value defined by Eq. (3.19). If the required accuracy is not too high one can use Eq. (3.19) even for $l \geq 3$. This leads to the result

$$E^{(\mathcal{D})} + E^{(\mathcal{N})} = -\frac{1}{a}\,0.220958\ldots. \tag{3.20}$$

Hence, taking into account (3.7) we have for the Casimir energy of massless scalar field obeying the Neumann boundary conditions on the sphere

$$E^{(\mathcal{N})} = -\frac{1}{a}\,0.223777\ldots \tag{3.21}$$

As far as we know this result is obtained here for the first time.

IV. CONCLUSION

The direct summation of eigenfrequencies by calculation of the Casimir effect for nonflat boundaries (specifically for sphere) has been used only in pioneer paper by Boyer [18]. The fact that done by us is actually a development and maximum simplification of the Boyer method and bringing it to such a form when numerical calculations are practically not required (see Eq. (2.39)), and, what is more important, cut-off functions are not used. In other

approaches, for example, by making use of the Green function formalism [1,16], transition to the imaginary frequencies is used without detailed justification. Contour integration in the mode summation method supplies a clear explanation for this technical trick.

Let us turn now to removing the divergences in the problem under consideration. Once renormalization of the Casimir energy by the formula (2.1) is accomplished the divergence for all that remains. In the general case the formula concerned has the form

$$E = \frac{C_1 + C_2^\infty}{a}, \tag{4.1}$$

where C_1 is finite constant, and C_2 is a divergent expression. Thus for example, for the electromagnetic field $C_1 = 0.046176\ldots$ and C_2^∞ is given by the divergent series

$$C_2^\infty = -\frac{3}{64}\sum_{l=1}^{\infty}(l+1/2)^0. \tag{4.2}$$

To remove this divergence we have applied the formal technique of the zeta function renormalization. In Ref. [1] the finite result was obtained here by making use of an exponential cutting multiplier splitting the arguments of the field operators in energy-momentum tensor. When calculating the Casimir energy for scalar massless field obeying the Dirichlet boundary conditions on sphere $C_1 = 0.002819\ldots$ and C_2^∞ stands for the sum of the divergent series

$$C_2^\infty = -\frac{1}{2}\sum_{l=0}^{\infty}(l+1/2)^2 - \frac{1}{128}\sum_{l=0}^{\infty}(l+1/2)^0. \tag{4.3}$$

Both in our paper and in Ref. [16] this divergence has been taken away by formal technique of the Hurwitz ζ-function.

Here the following question arises: to renormalization of which parameter does the removal of the divergence C_2^∞ correspond? After renormalization of the Casimir energy according to Eq. (2.1) only one parameter, namely, radius of the sphere a, is available. Therefore it is natural to treat the removal of the divergence C_2^∞ as the renormalization of the sphere radius. This can be done in a standard way by transition from the initial (bare) radius a to the physical (observable) radius a_{phys}: $a = a_{phys} + \delta a$, where δa is the appropriate counterterm. In view of this, Eq. (4.1) can be rewritten as

$$E = \frac{C_1 + C_2^\infty}{a} = \frac{C_1 + C_2^\infty}{a_{phys} + \delta a} = \frac{C_1}{a_{phys}}\frac{(1 + C_2^\infty/C_1)}{(1 + \delta a/a_{phys})}. \tag{4.4}$$

Setting

$$\frac{\delta a}{a_{phys}} = \frac{C_2^\infty}{C_1}, \tag{4.5}$$

one arrives at the finite result

$$E = \frac{C_1}{a_{phys}}. \tag{4.6}$$

This reasoning seems to be more consistent as compared with, for example, introduction into consideration of a phenomenological interaction [8] localized on sphere with subsequent renormalization of the coupling constant of this interaction that finally takes up divergence C_2^∞. It stands no reason that explanation suggested is also applicable to the calculation of the Casimir effect for the field confined inside the cavity. Certainly, in this case the values of the constants C_1 and C_2^∞ in Eq. (4.1) will be different.

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TABLE I. $\bar{Q}_l$ obtained by numerical integration according to Eqs. (3.2), (3.5), and asymptotic formula (3.6) for $\bar{Q}_l^{asym}$.

l	$\bar{Q}_l$	$\bar{Q}_l^{asym}$
0	0.001913	0.004273
1	0.000398	0.000474
2	0.000159	0.000171
3	0.000084	0.000087
4	0.000052	0.000053

TABLE II. With increasing l $\bar{P}_l$ rapidly approaches to its asymptotic value $\bar{P}_l^{asym}$ given in (3.19).

l	$\bar{P}_l$	$\bar{P}_l^{asym}$
0	-0.211491	-0.037351
1	-0.004800	-0.004150
2	-0.001582	-0.001494
3	-0.000775	-0.000762